Treasure Hunt
Midpoints and Bisectors

When you hear the phrase “treasure hunt,” you may think of pirates, buried treasure, and treasure maps. However, there are very few documented cases of pirates actually burying treasure, and there are no historical pirate treasure maps! So where did this idea come from?

Robert Louis Stevenson’s book *Treasure Island* is a story all about pirates and their buried gold, and this book greatly influenced public knowledge of pirates. In fact, it is Stevenson who is often credited with coming up with the concept of the treasure map and using an X to mark where a treasure is located.

Have you ever used a map to determine your location or the location of another object? Did you find it difficult or easy to use? How does the idea of a treasure map relate to a familiar mathematical concept you are very familiar with?
Problem 1

The map of a playground situated in the first quadrant of the coordinate plane is used to locate various items. Students will answer questions by determining the coordinates of points describing vertical or horizontal distances and their respective midpoints.

Grouping

- Ask a student to read the introduction. Discuss the context and complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

- Do the merry-go-round and the slide have a horizontal or vertical relationship?
- Do the slide and the swings have a horizontal or vertical relationship?
- Did you have to use the Distance Formula to determine the distance between the merry-go-round and the slide? Explain.

PROBLEM 1 Locating the Treasure

Ms. Lopez is planning a treasure hunt for her kindergarten students. She drew a model of the playground on a coordinate plane as shown. She used this model to decide where to place items for the treasure hunt, and to determine how to write the treasure hunt instructions. Each grid square represents one square yard on the playground.

1. Ms. Lopez wants to place some beads in the grass halfway between the merry-go-round and the slide.
   a. Determine the distance between the merry-go-round and the slide. Show all your work.
      
      \[11 - 3 = 8\]
      The merry-go-round and the slide are 8 yards apart.
   
   b. How far should the beads be placed from the merry-go-round and the slide?
      The beads should be placed 4 yards from the slide and 4 yards from the merry-go-round.
   
   c. Write the coordinates for the location exactly halfway between the merry-go-round and the slide. Graph a point representing the location of the beads on the coordinate plane.
      The coordinates for the location exactly halfway between the merry-go-round and the slide are (7, 2).
   
   d. How do the x- and y-coordinates of the point representing the location of the beads compare to the coordinates of the points representing the locations of the slide and the merry-go-round?
      The y-coordinates of all three points are the same.
      The x-coordinate of the point representing the beads is 4 units greater than the x-coordinate of the point representing the slide and is 4 units less than the x-coordinate of the point representing the merry-go-round.

Remember to include units when describing distances.
2. Ms. Lopez wants to place some kazoos in the grass halfway between the slide and the swings.
   a. Write the coordinates for the location of the kazoos. Graph the location of the kazoos on the coordinate plane.
      The distance between the swings and the slide is $12 - 2$, or 10 yards. So, the kazoos are 5 yards from the swings and 5 yards from the slide.
      The coordinates of the point representing the location of the kazoos are $(3, 7)$.
   b. How do the $x$- and $y$-coordinates of the point representing the location of the kazoos compare to the coordinates of the points representing the locations of the slide and the swings?
      The $x$-coordinates for all three points are the same.
      The $y$-coordinate of the point representing the kazoos is 5 units greater than the $y$-coordinate of the point representing the slide and is 5 units less than the $y$-coordinate of the point representing the swings.

3. Ms. Lopez wants to place some buttons in the grass halfway between the swings and the merry-go-round.
   a. Determine the distance between the swings and the merry-go-round.
      \[
      d = \sqrt{(3 - 11)^2 + (12 - 2)^2} \\
      = \sqrt{(-8)^2 + 10^2} \\
      = \sqrt{64 + 100} \\
      = \sqrt{164} \\
      \approx 12.8
      \]
      The swings and the merry-go-round are about 12.8 yards apart.
   b. How far should the buttons be placed from the swings and the merry-go-round?
      The buttons should be placed about 6.4 yards from the swings and about 6.4 yards from the merry-go-round.
   c. How is determining the coordinates for the location of the buttons different than determining the coordinates for the locations of the beads or the kazoos?
      The beads were along a horizontal line with the slide and the merry-go-round so its $y$-coordinate was the same.
      The kazoos were along a vertical line with the slide and the swings so its $x$-coordinate was the same.
      The $x$- and $y$-coordinates of the buttons are both different than the $x$- and $y$-coordinates of the swings and the merry-go-round.
**Grouping**

Have students complete Questions 4 through 7 with a partner. Then have students share their responses as a class.

**Guiding Questions for Share Phase, Questions 4 through 7**

- How did you determine the half-way point between the slide and the swings?
- Why are the $x$-coordinates of the kazoos, the slide, and the swing the same value?
- Why aren’t the $y$-coordinates of the kazoos, the slide, and the swing the same value?
- How is calculating the location of the pile of buttons different than calculating the location of the beads or kazoos in the previous questions?
- What geometric figure is formed when the locations of the swings, slide, and merry-go-round are connected?

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**d.** Write the coordinates for the location of the kazoos. Graph the location of the buttons on the coordinate plane.

The coordinates of the point representing the location of the kazoos are $(7, 7)$.

Suppose the slide, the swings, and the merry-go-round were at different locations but oriented in a similar manner. You can generalize their locations by using $x_1, x_2, y_1,$ and $y_2,$ and then solve for the distances between each using variables.

**4.** Use the diagram to describe each distance algebraically.

a. the vertical distance from the $x$-axis to the slide

The vertical distance to the slide is $y_1$.

b. the distance from the slide to the swings

The distance from the slide to the swings is $y_2 - y_1$.

c. half the distance from the slide to the swings

Half the distance from the slide to the swings is $\frac{y_2 - y_1}{2}$.

d. the vertical distance from the $x$-axis to the slide plus half the distance from the slide to the swings

The vertical distance from the $x$-axis to the slide plus half the distance from the slide to the swings is $y_1 + \frac{y_2 - y_1}{2}$.
Guiding Questions for Discuss Phase

- Why must $y_1$ be added when writing the expression to determine the distance between the two points?
- Why must $x_1$ be added when writing the expression to determine the distance between the two points?
- What common denominator can be used to combine the terms in the expression?
- Can this midpoint formula be used with any two points on the coordinate plane? Explain.
- Why do you think division by two is used in the midpoint formula?
- What is a good way to remember the midpoint formula?

5. Simplify your expression from Question 4, part (d).

$$y_1 + \frac{y_2 - y_1}{2} = \frac{2y_1 + y_2 - y_1}{2} = \frac{2y_1 + y_2 - y_1}{2} = y_1 + y_1$$

6. Use the diagram to describe each distance algebraically.
   a. the horizontal distance from the y-axis to the slide
      The horizontal distance to the slide is $x_1$.
   b. the distance from the slide to the merry-go-round
      The distance from the slide to the merry-go-round is $x_2 - x_1$.
   c. half the distance from the slide to the merry-go-round
      Half the distance from the slide to the merry-go-round is $\frac{x_2 - x_1}{2}$.
   d. the horizontal distance from the y-axis to the slide plus half the distance from the slide to the merry-go-round
      The horizontal distance from the y-axis to the slide plus half the distance from the slide to the merry-go-round is $x_1 + \frac{x_2 - x_1}{2}$.

7. Simplify your expression from Question 6, part (d).

$$x_1 + \frac{x_2 - x_1}{2} = \frac{2x_1 + x_2 - x_1}{2} = \frac{2x_1 + x_2 - x_1}{2} = \frac{x_1 + x_2}{2}$$

The coordinates of the points that you determined in Questions 5 and 7 are midpoints. A midpoint is a point that is exactly halfway between two given points. The calculations you performed can be summarized by the Midpoint Formula.

The Midpoint Formula states that if $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

8. Use the Midpoint Formula to determine the location of the buttons from Question 3.

The swings are at $(3, 12)$ and the merry-go-round is at $(11, 2)$. The buttons are halfway between the swings and the merry-go-round.

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{3 + 11}{2}, \frac{12 + 2}{2} \right) = \left( \frac{14}{2}, \frac{14}{2} \right) = (7, 7)$$

The buttons are located at $(7, 7)$. 
Note
Subtracting a negative number is often a difficult concept. Students are not always aware of why they are changing the signs, they just memorized the manipulation. An example of why this happens may be meaningful.
Problem 2
Scenarios are given asking students to determine a location \( \frac{1}{2} \), \( \frac{1}{3} \), and \( \frac{1}{4} \) of the distance between two specified points on the coordinate plane. Students use the Midpoint Formula and determine when it is useful to solve similar situations.

Grouping
Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4
- How is this scenario different from the playground scenario?
- Is there more than one way to solve this problem?
- How would your solving this problem be different if the key was located \( \frac{3}{4} \) of the way between the back door and the oak tree?
- How did you locate Jean’s key when it was buried \( \frac{1}{4} \) of the way between the front porch and the rose bush?
- Why couldn’t you locate Jean’s key when it was buried \( \frac{1}{3} \) of the way between the front porch and the rose bush?
- In which situations was the midpoint formula helpful?
- In which situations was the midpoint formula not helpful?

PROBLEM 2 Jack’s Spare Key

1. Jack buried a spare key to his house in the backyard in case of an emergency. He remembers burying the key halfway between the back door and an oak tree. The location of the back door is point \( B(2, 3) \), and the location of the oak tree is point \( T(12, 3) \).
   a. Determine the location of the key. Show all of your work. Then graph the location of the key as point \( M \).

   \[
   \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
   \]
   \[
   \left( \frac{2 + 12}{2}, \frac{3 + 3}{2} \right) = \left( \frac{14}{2}, \frac{6}{2} \right) = (7, 3)
   \]
   The key is located halfway between point \( B \) and point \( T \) at point \( M(7, 3) \).

   b. Suppose Jack buried his spare key \( \frac{1}{3} \) of the way between the back door and the oak tree. Determine the location of the key. Show all of your work.

   \[
   d = \sqrt{(12 - 2)^2 + (3 - 3)^2}
   \]
   \[
   d = \sqrt{10^2 + 0^2}
   \]
   \[
   d = \sqrt{100 + 0}
   \]
   \[
   d = 10
   \]
   The distance between the back door and the oak tree is 10 units. The distance between the back door and the key is \( \frac{10}{3} \) units. The \( x \)-coordinate of the key is \( 2 + \frac{10}{3} \), or \( \frac{16}{3} \). The \( y \)-coordinate is 3.
   The key is located \( \frac{1}{3} \) of the way between point \( B \) and point \( T \) at \( \left( \frac{16}{3}, 3 \right) \).
2. Jean also buried her house key. She remembers burying the key between the front porch and a rose bush. The location of the front porch is point \( P(1, 2) \) and the location of the rose bush is point \( B(16, 14) \).

a. Suppose Jean buried her key \( \frac{1}{2} \) of the way between the front porch and the rose bush. Determine the location of the key. Show all your work.

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{1 + 16}{2}, \frac{2 + 14}{2} \right)
\]

\[
= \left( \frac{17}{2}, \frac{16}{2} \right)
\]

\[
= \left( 8.5, 8 \right)
\]

The point \( (8.5, 8) \) is halfway between the front porch and the rose bush.

b. Suppose Jean buried her key \( \frac{1}{4} \) of the way between the front porch and the rose bush. Determine the location of the key. Show all your work.

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{1 + 8.5}{2}, \frac{2 + 8}{2} \right)
\]

\[
= \left( \frac{9.5}{2}, \frac{10}{2} \right)
\]

\[
= \left( 4.75, 5 \right)
\]

The point \( (4.75, 5) \) is one-fourth of the way between the front porch and the rose bush.

c. Suppose Jean buried her key \( \frac{1}{3} \) of the way between the front porch and the rose bush. Explain why the Midpoint Formula is not helpful in determining the location of Jean’s spare key.

The Midpoint Formula is only helpful when I want to locate a point that is halfway between two other points. It will not locate a point that is one-third of the way between two points.
3. Rick and Courtney used different methods to determine the location that is $\frac{1}{3}$ of the way between the front porch and the rose bush.

**Rick**

I drew a vertical line from point B and a horizontal line from point P and determined the coordinates of point Z where they intersected. I divided the horizontal line and the vertical line into 3 equal parts and wrote the coordinates. I used those coordinates to divide segment EP into three parts. Then, I calculated the coordinates along EP.

**Courtney**

I thought about the slope of segment BP. If I want to divide segment BP into three parts, then I need to start at point P and perform three vertical and horizontal shifts to get to point B. So, I can just divide the total horizontal and vertical shifts by three to calculate each vertical and horizontal shift. After that, determining the coordinates is a snap!

Calculate the location of the point that is $\frac{1}{3}$ of the way between the front porch and the rose bush using Rick’s method and Courtney’s method. Use a graph to support your work.

a. Rick’s Method

The coordinates of point Z are (16, 2).

Segment BZ is 14 – 2, or 12 units. Dividing segment BZ into three equal segments means that each segment is 4 units long. So, the coordinates are (16, 6) and (16, 10).

Segment PZ is 16 – 1, or 15 units. Dividing segment PZ into three equal segments means that each segment is 5 units. So, the coordinates are (6, 2) and (11, 2).

The coordinates along BP are (6, 6) and (11, 10). The point (6, 6) is one-fourth of the way between the front porch and the rose bush.
b. Courtney's Method

The vertical distance between point $P$ and point $B$ is $14 - 2$, or 12 units. So, I need three vertical shifts of 4 units to move from point $P$ to point $B$.

The horizontal distance between point $P$ and point $B$ is $16 - 1$, or 15 units. So, I need three horizontal shifts of 5 units to move from point $P$ to point $B$.

Starting at point $P$, the coordinates along segment $BP$ are $(6, 6)$ and $(11, 10)$.

4. Suppose Jean buried her key $\frac{1}{2}$ of the way between the front porch and the rose bush. Determine the location of the key. Show all your work.

The vertical distance between point $P$ and point $B$ is $14 - 2$, or 12 units. So, I need five vertical shifts of 2.4 units to move from point $P$ to point $B$.

The horizontal distance between point $P$ and point $B$ is $16 - 1$, or 15 units. So, I need three horizontal shifts of 3 units to move from point $P$ to point $B$.

Starting at point $P$, the coordinates along segment $BP$ are $(4, 4.4)$, $(7, 6.8)$, $(10, 9.2)$, and $(13, 11.6)$. 

Problem 3
Students are provided with the steps for locating a midpoint by bisecting a line segment using construction tools. Segment bisector is defined. Students first use patty paper to bisect a line segment then use a compass and straightedge. Students will then locate the midpoint of several line segments using construction tools. Each line segment is drawn in a different direction.

Grouping
- Ask a student to read the definitions and example. Discuss the context and complete Questions 1 and 2 as a class.
- Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase
- If the line segment was vertically drawn on the patty paper, how would you perform the construction?
- Why does this construction work?
- How wide should the compass be opened to strike the arcs?
- Why does this construction work?
- What is the significance of the location where two arcs intersect?

PROBLEM 3  Stuck in the Middle
In the previous problem, you located the midpoint of a line segment on the coordinate plane. The lengths of the line segments on the plane are measurable.

In this problem, you will locate the midpoint of a line segment when measurement is not possible. This basic geometric construction used to locate a midpoint of a line segment is called **bisecting a line segment**. When bisecting a line segment, you create a segment bisector. **A segment bisector** is a line, line segment, or ray that divides a line segment into two line segments of equal measure, or two congruent line segments.

Just as with duplicating a line segment, there are a number of methods to bisect a line segment. You can use tracing paper—also known as patty paper—to bisect a line.

1. Use tracing paper to duplicate a line segment. How do you know your bisector and midpoint are accurate?

   When I folded the paper, I made sure the endpoints were aligned. This means the crease must be exactly in the middle of the line.
2. Thomas determined the midpoint of \( AB \) incorrectly.

Thomas just folded the paper in half which did not result in a correct midpoint. I can tell this is incorrect because the bisector did not divide the line segment into two equal parts.

To correctly determine the midpoint, Thomas must make sure that when he folds the paper the endpoints line up.

You can use a compass and straightedge to construct a segment bisector.

Line \( EF \) bisects line segment \( AB \). Point \( M \) is the midpoint of line segment \( AB \).
Guiding Questions for Share Phase, Questions 3 through 6

- How does the direction the line segments are drawn (vertical, horizontal, slanted) affect the construction of the midpoint?
- Do all lines have midpoints? How can this be proven?
- Do you prefer drawing the first arc using the left or the right endpoint of the line segment? Why?
- How can a compass be used to verify a line segment has been bisected?

3. Aaron is determining the midpoint of line segment RS. His work is shown.

He states that because the arcs do not intersect, this line segment does not have a midpoint. Kate disagrees and tells him he drew his arcs incorrectly and that he must redraw his arcs to determine the midpoint. Who is correct? Explain your reasoning.

Kate is correct. All line segments have midpoints. If the arcs do not intersect, Aaron should open his compass further and redraw the arcs.

4. Use construction tools to locate the midpoint of each given line segment. Label each midpoint as M.

a.
b. 

\[ \begin{align*} 
C & \quad D \\
M & \\
\end{align*} \]

c. 

\[ \begin{align*} 
F & \quad G \\
M & \\
\end{align*} \]
5. Perform each construction shown. Then explain how you performed each construction.
   a. Locate a point one-fourth the distance between point A and point B.

   ![Diagram showing construction of a point one-fourth the distance between A and B]

   Point P is \( \frac{3}{4} \) the way between point A and point B. First, I constructed the midpoint M of AB. Point M is one-half the distance between points A and B. Then I constructed the midpoint P of AM. Point P is one-fourth the distance between points A and B.

   b. Locate a point one-third the distance between point A and point B.

   ![Diagram showing construction cannot be done based on what students know at this point in the course]

   This construction cannot be done based on what students know at this point in the course.

6. Explain how you can duplicate a line segment to verify that the midpoint resulting from bisecting the line segment is truly the midpoint of the segment.

   I can use my compass to copy each line segment formed by bisecting the line segment to see if they map onto themselves.
Talk the Talk

In this lesson, students explored how to locate the midpoint of a line segment using the Midpoint Formula and by construction using construction tools. Students will state the similarities and differences of using the two methods to locate the midpoint of a line segment.

Grouping

Have students complete the Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

1. When bisecting a line segment using construction tools, does it make a difference which endpoint you use to draw the first arc?
   No. It does not make a difference which endpoint I use to draw the first arc. The results are the same.

2. When locating the midpoint of a line segment on a coordinate plane using the Midpoint Formula, does it make a difference which endpoint you use as \( x_1 \) and \( y_1 \)?
   No. It does not make a difference which endpoint I use as \( x_1 \) and \( y_1 \). The results are the same.

3. How will you decide if you should use the Midpoint Formula or construction tools to locate a midpoint?
   If I know the coordinates of the endpoints of the line segment, then I can use the Midpoint Formula. If I do not know the coordinates of the endpoints, I must use construction tools.

Be prepared to share your solutions and methods.