3.2 Area and Perimeter of Triangles on the Coordinate Plane

Bell Work:

1. Find the lengths of the 3 sides of the triangle. Use the distance formula if necessary.

   \[ AB = \sqrt{(0 - (-6))^2 + (4 - 0)^2} = \sqrt{36 + 16} = 2\sqrt{13} \]

   \[ AC = \sqrt{(8 - 0)^2 + (0 - 4)^2} = \sqrt{64 + 16} = 4\sqrt{5} \]

   \[ BC = 8 - 6 = 14 \]

2. The perimeter of a triangle is the sum of measures of its 3 sides.

   Find the perimeter of triangle ABC.

   Perimeter is about 51.78 units

3. Find \( AD \), the height of the triangle.

   \( h = 4 \)

4. The area of a triangle is \( \frac{1}{2} \) times the base times the height.

   Find the area of the triangle.

   \[ A = \frac{1}{2} (14)(4) = 28 \text{ square units} \]

Triangle ABC has vertices A\((-5, 2)\), B\((1, 5)\) and C\((2, 2)\).

Plot the points on the graph and draw the triangle.

1. What is the perimeter of the triangle?

   \[ \text{length } AB = \sqrt{(1 - (-5))^2 + (5 - 2)^2} = \sqrt{36 + 9} = 3\sqrt{5} \]

   \[ \text{length } BC = \sqrt{(2 - 1)^2 + (2 - 5)^2} = \sqrt{1 + 9} = \sqrt{10} \]

   \[ \text{length } AC = 7 \quad \text{Perimeter = about 16.87 units} \]

2. Draw the height of the triangle.

   Remember that the height must be drawn perpendicular to the base.

   Label the point on the base where the height meets the base as D. \( D(1, 2) \)

3. To be assured that the height is drawn perpendicular to the base find the slope of the height and the slope of the base. They should be opposite reciprocals.

   slope \( BD = \text{undefined} \quad \text{slope AC} = 0 \)
4. Find the area of the triangle. Area of a triangle is $\frac{1}{2}$ bh, you need to find the lengths of BD and AC.

- length BD = 3
- length AC = 7

Area triangle ABC = $\frac{1}{2}$ bh = $\frac{1}{2}$ (7)(3) = 21/2 = 10.5 square units

By translating the triangle we might make these calculations easier. Arithmetic with 0’s is always easier than arithmetic with other numbers. Translate the triangle so that one or more of the vertices have a 0 in the ordered pair.

**Recall:** translating is sliding up/down and/or right/left. How much and in what direction should we translate triangle ABC?

Label the translated triangle A'B'C'?

- A'(-6, 0)  B'(0, 3)  C'(1, 0)

What is the perimeter of triangle A'B'C'?

- length A'B' = $\sqrt{(0 - -6)^2 + (3 - 0)^2} = \sqrt{36 + 9} = 3\sqrt{5}$
- length B'C' = $\sqrt{(1 - 0)^2 + (0 - 3)^2} = \sqrt{1 + 9} = \sqrt{10}$
- length A'C' = 7

perimeter A'B'C' = about 16.87 units

What is the area of triangle A'B'C'?

- Draw the height B'D'.
- D'(0, 0)

What is this height?

- length B'D' = 3

A = $\frac{1}{2}$ bh = $\frac{1}{2}$ (7)(3) = 10.5 square units

Since translation preserves segment lengths, it will also preserve area measure.
Finding the perimeter and area of A'B'C' should have been easier than finding the perimeter and area of triangle ABC.

One reason these problems were easy to do was that the base and height of the triangle were horizontal and vertical segments.
The problem becomes more difficult if the base and height are not horizontal and vertical segments. Triangle ABC has vertices A(1, 2), B(3, 5), and C( 5, -1).

1. Graph the triangle on the grid.

2. Translate the triangle so that some of the coordinates have zeros. Call the triangle A’B’C’.

   \[ A'(0, -2) \quad B'(2, 0) \quad C'(4, -5) \]

3. Find the perimeter of triangle A’B’C’.

   \[
   \text{length } A'B' = \sqrt{(2 - 0)^2 + (0 - (-2))^2} = \sqrt{4 + 4} = 2\sqrt{2}
   \]

   \[
   \text{length } B'C' = \sqrt{(4 - 2)^2 + (-5 - 0)^2} = \sqrt{4 + 25} = \sqrt{29}
   \]

   \[
   \text{length } A'C' = \sqrt{(4 - 0)^2 + (-5 - (-2))^2} = \sqrt{16 + 9} = 5
   \]

   perimeter triangle A’B’C’ = about 13.21 units

Recall: Translation preserves segment lengths so the perimeter of triangle ABC will be the same as the perimeter of triangle A’B’C’.

4. To find the area of a triangle it is necessary to choose which side will be the base, because the height must be drawn perpendicular to the base from the vertex opposite the base. Any of the 3 sides of this triangle can be the base.

   base = A’B’

5. Draw the height, label the point where the height meets the base D’.
   Be careful, it is not so obvious where D’ should be or what its coordinates are.

7. To find the length of the height we will have to do the distance from a point to a line. This will be a 4 step process.
   a. Find the equation of the base. You know the endpoints of the base so you can use these to find the slope of the base. Then choose 1 of the two endpoints and use this and the slope in the point slope formula, \( y - y_1 = m(x - x_1) \). Simplify your answer to slope-intercept form (y = mx + b).

   \[ A'(0, -2) \quad C'(4, -5) \]

   \[
   m = \frac{(-5 - (-2))/(4 - 0)} = -\frac{3}{4}
   \]

   Since A’ is the y-intercept, b = -2.

   \[
   y = -\frac{3}{4} x - 2
   \]
b. Find the equation of the height. The height is perpendicular to the base through the point opposite the base. You are finding the equation of a line perpendicular to a given line through a given point. The given line is the equation from (a) above and the given point is the point opposite the base. Use the point-slope formula: \( y - y_1 = m(x - x_1) \). Simplify your answer to slope-intercept form \( y = mx + b \).

Since the base has a slope of \(-3/4\), the height will have a slope of \(4/3\) (its negative reciprocal).
The height also passes through point \(B'(2, 0)\).

\[
y - 0 = 4/3(x - 2) \\
y = 4/3 x - 8/3
\]

c. Find the solution to the system of equations found in (a) and (b) above. This answer is the ordered pair of point \(D'\).

\[
y = -3/4 x - 2 \quad \text{and} \quad y = 4/3 x - 8/3 \\
-3/4 x - 2 = 4/3 x - 8/3 \quad \text{multiply both sides by 12 to clear the fractions} \\
12(-3/4 x - 2) = 12(4/3 x - 8/3) \\
-9x - 24 = 16x - 32 \quad y = -3/4(0.32) - 2 = -2.24 \\
+9x + 32 +9x + 32 \\
8 = 25x \quad (0.32, -2.24) \\
X = 0.32
\]

d. Use the distance formula to find the length of the height now that you know both of its endpoints.

Find the distance between \((0.32, -2.24)\) and \((2, 0)\).

\[
\sqrt{(2 - 0.32)^2 + (0 - -2.24)^2} = \sqrt{2.8224 + 5.0176} = \text{about 1.29}
\]

8. Find the area of triangle \(A'B'C'\).

\[
\text{area triangle ABC} = \text{area triangle A'B'C'} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} (5)(1.29) = \text{about 8.225 square units}
\]