In this lesson, you will:

- Rotate two-dimensional plane figures to generate three-dimensional figures.
- Give an informal argument for the volume of cones and pyramids.

Imagine that you are, right now, facing a clock and reading the time on that clock—let’s imagine that it’s 2:28.

Now imagine that you are blasted away from that clock at the speed of light, yet you are still able to read the time on it (of course you wouldn’t be able to, really, but this is imagination!).

What time would you see on the clock as you traveled away from it at the speed of light? The light bouncing off the clock travels at the speed of light, so, as you travel farther and farther away from the clock, all you could possibly see was the time on the clock as it was right when you left.

Einstein considered this “thought experiment” among many others to help him arrive at groundbreaking theories in physics.
Problem 1
Rotating a rectangle about an axis and stacking congruent discs are two methods used for creating the image of a cylinder. Students determine the area of an average disc which leads to the volume formula for any cylinder.

Grouping
Ask students to read introduction and worked example. Discuss Questions 1 and 2 as a class. Discuss as a class.

Guiding Questions for Discuss Phase
• How are the formulas for the area of a circle and volume of a cylinder related?
• What is the height of each disc if there are 10 discs and the total height is represented by $h$?
• What is the area of each disc in the cylinder?
• Are all discs in the cylinder congruent?
• What do the variables in the diagram represent?

PROBLEM 1 Building Cylinders
You have learned that when two-dimensional shapes are translated, rotated, or stacked, they can form three-dimensional solids. You can also use transformations and stacking to build formulas for three-dimensional figures.

1. Determine the volume for the cylinder shown. Show your work.

To calculate the volume of the cylinder, I can use the formula $V = \pi r^2 h$.
The volume is $\pi (8^2)(24)$, or approximately 4825.49 cubic centimeters.

To derive the formula for the volume of a cylinder, you can think of the cylinder as an infinite stack of discs, each with an area of $\pi r^2$. These discs are stacked to a height of $h$, the height of the cylinder.
So, the volume of the cylinder is $\pi r^2 \times h$, or $\pi r^2 h$.

2. Can you choose any disc in the cylinder and multiply the area by the height to calculate the volume of the cylinder? Explain your reasoning.
Yes. Every disc that makes up the cylinder is congruent. So, you can choose any disc and multiply its area by the height to calculate the volume of the cylinder.
Another way to think about a cylinder is the rotation of a rectangle about one of its sides as shown. The rotation of the set of points that make up the rectangle forms the cylinder.

To determine the volume of the cylinder, you can multiply the area of the rectangle by the distance that the points of the rectangle rotate. However, the points of the rectangle don’t all rotate the same distance. Consider a top view of the cylinder. The distance that point A rotates is greater than the distance that point B rotates.

You can’t calculate the distance that each point rotates because there are an infinite number of points, each rotating a different distance. But you can use an average, or typical point, of the rectangle.

3. Consider the dot plot shown.

a. What is the median of the data?
   The median of the data is the middle value when the data are ordered from least to greatest or greatest to least. The median of these data is 5.

b. Describe how you could determine the median of these data without doing any calculation.
   There are 11 values that are evenly spaced from 0 to 10, so the median should be the exact center value of the data set. There are five values to the left of 5 and five values to the right of 5, so 5 is the median of these data.

Remember that the median is a measure of center of a data set.
Grouping
Have students complete Questions 4 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 8
• How is the rotating rectangle related to the area of the cylinder?
• Do you prefer the infinite stack of discs or the rotating for deriving the volume formula for a cylinder? Explain.

4. What is the location of the average, or typical, point of the rectangle in terms of the radius and the height? Explain your reasoning.
   The average, or typical, point is the center point of the rectangle. The center of the width of the rectangle is $\frac{1}{2}r$. The center of the height of the rectangle is $\frac{1}{2}h$. So, the average, or typical, point of the rectangle is located at $\left(\frac{1}{2}r, \frac{1}{2}h\right)$.

5. What is the area of the rectangle that is rotated?
   The area of the rectangle is $rh$.

6. Use the average point of the rectangle to calculate the average distance that the points of the rectangle rotate. Explain your reasoning.
   Because the average point of the rectangle is located at $\frac{1}{2}r$, the average distance that all of the points of the rectangle are rotated is the circumference of a circle with a radius of $\frac{1}{2}r$.
   
   Circumference = $2\pi\left(\frac{1}{2}r\right)$
   = $\pi r$

   The average distance that all of the points of the rectangle are rotated is $\pi r$.

7. To determine the volume of the cylinder, multiply the area of the rectangle by the average distance that the points of the rectangle rotate. Calculate the volume of the cylinder.
   To determine the volume formula, I multiply the area of the rectangle, $rh$, by the average distance that all of the points of the rectangle are rotated, $\pi r$.
   $\pi r(h) = \pi rh$

8. Compare the volume that you calculated in Question 2 to the volume that you calculated in Question 7. What do you notice?
   Both methods give the formula for the volume of a cylinder, $\pi rh$. 
Problem 2
Rotating a right triangle about an axis and stacking an infinite number of similar discs are two methods for creating the image of a cone. The area of an average disc is calculated in which the length of the radius is determined algebraically. This leads to the volume formula for any cone.

Grouping
- Ask students to read introduction and worked examples. Then discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

PROBLEM 2 Building Cones
You can also think about a cone as a rotation of a right triangle about one of its legs. You can use the same strategy of determining the average, or typical, point in a right triangle to derive the formula for the volume of a cone. The figure shown represents a cone formed by rotating a right triangle about one of its legs.

Previously, you explored points of concurrency of a triangle. Recall that the centroid is the point of concurrency of the three medians of a triangle. A median connects a vertex to the midpoint of the opposite side. You can use the centroid as the typical point of the triangle to derive the volume. The centroid of the right triangle is shown.
Let’s review how to calculate the coordinates of the centroid of triangle ABC.

Triangle ABC has vertices at A (0, 0), B (0, 18), and C (12, 0).

First, determine the locations of the midpoints of side AB and side AC.
- Midpoint of side AC: \( \left( \frac{0 + 12}{2}, \frac{0 + 0}{2} \right) \), or (6, 0)
- Midpoint of side AB: \( \left( \frac{0 + 0}{2}, \frac{0 + 18}{2} \right) \), or (0, 9)

Next, determine an equation representing each median.
- The slope of BE is \(-3\), and the y-intercept is 18.
  - Slope of BE: \( \frac{0 - 18}{0 - 6} = \frac{-18}{-6} = 3 \)
  - So, the equation of BE is \( y = 3x + 18 \).
- The slope of CD is \(\frac{3}{4}\), and the y-intercept is 9.
  - Slope of CD: \( \frac{0 - 9}{12 - 0} = \frac{-9}{12} = \frac{-3}{4} \)
  - So, the equation of CD is \( y = \frac{3}{4}x + 9 \).

Why do we need to draw only 2 median lines instead of all 3?
Solve the system of equations to determine the coordinates of the centroid.

\[
\begin{align*}
\frac{3}{4}x + 9 &= -3x + 18 \\
\frac{3}{4}x &= -3x + 9 \\
\frac{1}{4}x &= x - 3 \\
x &= 4
\end{align*}
\]

The centroid of the triangle is located at (4, 6).
Guiding Questions for Share Phase, Questions 1 through 5

- When computing the average radius, how many radii are considered?
- When computing the average height, how many heights are considered?
- Why is the average radius located where the two lines intersect?
- What is the slope formula?
- What is the slope-intercept form for a line?
- What are the two linear equations that determine the average length of a radius?
- What is the length of the radius of the entire cone?
- What is the length of the average radius?
- Is 4 one-third of 12?
- What is the slope and \( y \)-intercept of the first line?
- What is the slope and \( y \)-intercept of the second line?
- What fraction is associated with the radius of a typical disc?

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1. What is the area of triangle \( ABC \)?
   The base of the triangle is 12 units, and the height is 18 units, so the area of the triangle is \( \frac{1}{2}(12)(18) \), or 108 square units.

2. What is the average distance that all the points of the triangle are rotated? Show your work and explain your reasoning.
   Because the centroid has an \( x \)-coordinate of 4, the average distance that all the points of the rectangle are rotated is the circumference of a circle with a radius of 4. So, the average distance all the points are rotated is \( 2\pi(4) \), or \( 8\pi \) units.

3. Determine the volume of the cone. Show your work and explain your reasoning.
   The volume of the cone is the product of the area of the triangle, 108 square units, and the average distance all the points of the triangle are rotated, \( 8\pi \).
   \[ 108 \text{ square units} \times 8\pi \text{ units} = 864\pi \text{ cubic units} \]

4. Use the formula for the volume of a cone, \( \frac{1}{3}\pi r^2h \), to calculate the volume of the cone. Show your work.
   \[ \frac{1}{3}\pi(12)^2(18) = \frac{1}{3}\pi(2592) \]
   \[ = 864\pi \text{ cubic units} \]

5. Compare the volume that you calculated by rotating the right triangle to the volume that you calculated using the formula for the volume of a cone. What do you notice?
   The volume calculations result in the same volume.
Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- The centroid is the point of concurrency for the three medians of a triangle. But, the solution only shows two of the medians. Is there a mistake? Explain.
- If two different medians of the triangle were used, would it result in the same volume formula?
- Does it matter which two medians are used for deriving the formula of a cone?

6. Derive the formula for the volume of any cone with radius, \( r \), and height, \( h \), by rotating a right triangle with vertices at \((0, 0)\), \((0, h)\), and \((r, 0)\).

First, I determine the locations of the midpoints of side \(AB\) and side \(AC\).

- Midpoint of side \(AC\): \(\left[\frac{0 + r}{2}, \frac{0 + 0}{2}\right]\) or \(\left[\frac{r}{2}, 0\right]\)
- Midpoint of side \(AB\): \(\left[\frac{0 + 0}{2}, \frac{0 + h}{2}\right]\) or \(\left[0, \frac{h}{2}\right]\)

Next, I determine an equation representing each median.

- The slope of median \(CD\) is \(-\frac{h}{2r}\), and the \(y\)-intercept is \(\frac{h}{2}\).

\[\text{Slope of median } CD = \frac{0 - \frac{3h}{2}}{0 - 0} = -\frac{h}{2r}\]

So, the equation of median \(CD\) is \(y = -\frac{h}{2r}x + \frac{h}{2}\).

- The slope of median \(BE\) is \(-\frac{2h}{r}\), and the \(y\)-intercept is \(h\).

\[\text{Slope of median } BE = \frac{0 - h}{2r - 0} = -\frac{h}{2r}\]

So, the equation of median \(BE\) is \(y = -\frac{2h}{r}x + h\).

Then, I solve the system of equations to determine the coordinates of the centroid.

\[
\begin{align*}
-\frac{h}{2r}x + \frac{1}{2}h &= -\frac{2h}{r}x + h \\
\frac{2h}{r} - \frac{h}{2r}x &= \frac{1}{2}h \\
\frac{3h}{2r}y &= \frac{1}{2}h \\
x &= \frac{2r}{2} = \frac{r}{3} \\
y &= \frac{b}{3} + \frac{3h}{3}
\end{align*}
\]

The centroid of any triangle is located at \(\left(\frac{1}{3}, \frac{1}{3}\right)\).

The centroid of a triangle is located at \(\left(\frac{1}{3}, \frac{1}{3}\right)\), so the average distance all the points of a triangle are rotated to create a cone is \(2\pi(\frac{1}{3})\), or \(\frac{2}{3}\pi r\).

The area of the triangle is \(\frac{1}{2}bh\).

So, the volume of any cone is \(\frac{2}{3}\pi r \times \frac{1}{2}bh = \frac{2}{3}\pi r h\), or \(\frac{1}{3}\pi r h\).
Talk the Talk
Students write an informal argument to show how Cavalieri’s principle can be applied to determining the volume of a pyramid.

Grouping
Have students complete Questions 1–6 with a partner. Then have students share their responses as a class.

PROBLEM 3 And Now, Pyramids
You can use what you know about the similarities and differences between cylinders and cones to make conjectures about the volume of pyramids.

1. Compare different ways to create cylinders and cones. What similarities and differences are there between creating cylinders and cones:
   a. by stacking?
   Both solid figures are created by stacking circles. A cylinder is created by stacking congruent circles. A cone is created by stacking similar circles that are not congruent.
   b. by rotating?
   For both cylinders and cones, I determine an average point of the polygon that is rotated to create the solid figure. I use this average point to calculate the average distance that all of the points of the polygon are rotated. I then multiply the area of the polygon by this average distance to determine the volume formula.
   To create a cylinder, I rotate a rectangle with an average point located at \( \left( \frac{1}{2}r, \frac{1}{2}h \right) \).
   To create a cone, I rotate a triangle with an average point located at \( \left( \frac{1}{3}r, \frac{1}{3}h \right) \).

2. Analyze the formulas for the volumes of the cylinder and cone.
   - Volume of cylinder = \( \pi r^2 h \)
   - Volume of cone = \( \frac{1}{3} \pi r^2 h \)
   a. Which part of each formula describes the area of the base of each solid figure, \( B \)? Explain why.
      The expression \( \pi r^2 \) describes the area of the base of each solid figure, because the base of each solid is a circle. The area of a circle is given as \( A = \pi r^2 \).
   b. Which part of each formula describes the height of each solid figure?
      For each solid figure, the variable \( h \) in the volume formula represents the height.
   c. Rewrite the volume formulas for each solid figure using the variables \( B \) and \( h \).
      Cylinder: \( V = Bh \)
      Cone: \( V = \frac{1}{3}Bh \)
3. How are the formulas for the volumes of a cylinder and cone similar and different?
Both formulas involve multiplying the area of the base by the height. The volume formula for a cone is \( \frac{1}{3} \) the volume of a cylinder with the same base area and height, so I have to multiply \( Bh \) by \( \frac{1}{3} \) to obtain the volume of a cone.

4. Now compare different ways to create prisms and pyramids. What similarities and differences are there between creating prisms and pyramids:
   a. by stacking?
      Both solid figures are created by stacking polygons. A prism is created by stacking congruent polygons. A pyramid is created by stacking similar polygons that are not congruent.
   b. by rotating?
      Prisms and cylinder cannot be created using rotations.

5. Analyze the formula for the volume of a prism.
   Volume of prism: \( V = Bh \)
   a. Which part of the formula represents the area of the base?
      The variable \( B \) represents the area of the base.
   b. Which part of the formula represents the height?
      The variable \( h \) represents the height.

6. Based on your answers to Questions 1 through 5, what conjecture can you make about the formula for the volume of any pyramid? Explain your reasoning.
   I think the volume formula for a pyramid is \( \frac{1}{3} \) the volume of a prism with the same base area and height. A prism is created by stacking congruent polygons, which is similar to creating a cylinder by stacking congruent circles. A pyramid is created by stacking similar polygons that are not congruent, which is similar to creating a cone by stacking similar circles.

Be prepared to share your solutions and methods.